AP Calculus AB
3.7: P. 155 #3-42 (multiples of 3), 40, 41, 43, 45, 46, 48, 50

(3) $y = \sqrt[3]{x} = x^{\frac{1}{3}}$

$\frac{dy}{dx} = \frac{1}{3}x^{-\frac{2}{3}}$

$\frac{dy}{dx} = \frac{1}{3\sqrt[3]{x^2}}$

(6) $y = (1-6x)^{2/3}$

$\frac{dy}{dx} = \frac{2}{3}(1-6x)^{-\frac{1}{3}}(-6)$

$\frac{dy}{dx} = -\frac{4}{\sqrt[3]{1-6x}}$

(9) $x^2y + xy^2 = 6$

$\frac{d}{dx}\left[x^2y + xy^2\right] = \frac{d}{dx}[6]$

$\left[x^2\frac{dy}{dx} + 2xy\right] + \left[x \cdot 2y\frac{dy}{dx} + y^2\right] = 0$

$x^2\frac{dy}{dx} + 2xy\frac{dy}{dx} = -y^2 - 2xy$

$\frac{dy}{dx} = \frac{-y^2-2xy}{x^2+2xy} = \frac{-(y^2+2xy)}{x^2+2xy}$

(12) $x^2 = \frac{x-y}{x+y}$

$\frac{d}{dx}\left[x^2\right] = \frac{d}{dy}\left[\frac{x-y}{x+y}\right]$

$2x = \frac{(x+y)(1-\frac{dy}{dx}) - (x-y)(1+\frac{dy}{dx})}{(x+y)^2}$

$2x = \frac{\cancel{x} - x\frac{dy}{dx} + y - \cancel{y\frac{dy}{dx}} - \cancel{x} - x\frac{dy}{dx} + y + \cancel{y\frac{dy}{dx}}}{(x+y)^2}$

$2x(x+y)^2 = 2y - 2x\frac{dy}{dx}$

$x(x+y)^2 = y - x\frac{dy}{dx}$

$x\frac{dy}{dx} = y - x(x+y)^2$

$\frac{dy}{dx} = \frac{y - x(x+y)^2}{x}$

(15) $y = 3(\csc x)^{3/2}$

$\frac{dy}{dx} = \frac{9}{2}(\csc x)^{1/2}(-\csc x \cot x)$

$\frac{dy}{dx} = -\frac{9}{2}(\cot x)(\csc x)^{3/2}$

(18) $x = \sin y$

$\frac{d}{dx}[x] = \frac{d}{dx}[\sin y]$

$1 = (\cos y)\frac{dy}{dx}$

$\frac{dy}{dx} = \frac{1}{\cos y} = \sec y$

---

(21) $f''(x) = x^{-1/3}$

a) false b) true c) true d) true

---

(24) $x^{2/3} + y^{2/3} = 1$

$\frac{d}{dx}\left[x^{2/3} + y^{2/3}\right] = \frac{d}{dx}[1]$

$\frac{2}{3}x^{-1/3} + \frac{2}{3}y^{-1/3}\frac{dy}{dx} = 0$

$\frac{2}{3}y^{-1/3}\frac{dy}{dx} = -\frac{2}{3}x^{-1/3}$

$\frac{dy}{dx} = -\frac{x^{-1/3}}{y^{-1/3}}$

$\frac{dy}{dx} = -\frac{y^{1/3}}{x^{1/3}}$

$\boxed{\frac{dy}{dx} = -\sqrt[3]{\frac{y}{x}}}$

$\frac{dy}{dx} = -\left(\frac{y}{x}\right)^{1/3}$

$\frac{d^2y}{dx^2} = -\frac{1}{3}\left(\frac{y}{x}\right)^{-2/3}\left[\frac{x\frac{dy}{dx} - y}{x^2}\right]$

$\frac{d^2y}{dx^2} = \left(\frac{y}{x}\right)^{-2/3}\left[\frac{y - x\frac{dy}{dx}}{3x^2}\right]$

$\frac{d^2y}{dx^2} = \frac{x^{2/3}}{y^{2/3}}\left[\frac{y + x\left(\frac{y^{1/3}}{x^{1/3}}\right)}{3x^2}\right]$

$\frac{d^2y}{dx^2} = \frac{x^{2/3}\left[y + x^{2/3}y^{1/3}\right]}{3x^2 y^{2/3}}$

$\frac{d^2y}{dx^2} = \frac{y + x^{2/3}y^{1/3}}{3x^{4/3}y^{2/3}} = \frac{y^{1/3}\left(y^{2/3} + x^{2/3}\right)}{3x^{4/3}y^{2/3}}$

$\boxed{\frac{d^2y}{dx^2} = \frac{x^{2/3} + y^{2/3}}{3x^{4/3}y^{1/3}}}$

(27) $x^2 + xy - y^2 = 1, \quad (2,3)$

$\frac{d}{dx}\left[x^2 + xy - y^2\right] = \frac{d}{dx}[1]$

$2x + \left[x\frac{dy}{dx} + y\right] - 2y\frac{dy}{dx} = 0$

$(x - 2y)\frac{dy}{dx} = -(2x + y)$

$\frac{dy}{dx} = -\frac{(2x + y)}{x - 2y}$ At $(2,3)$, $\frac{dy}{dx} = -\frac{7}{-4} = \frac{7}{4}$

a) tangent line:

$y - 3 = \frac{7}{4}(x - 2)$

$y = \frac{7}{4}x - \frac{1}{2}$

b) normal line:

$y - 3 = -\frac{4}{7}(x - 2)$

$y = -\frac{4}{7}x + \frac{29}{7}$

---

(30) $y^2 - 2x - 4y - 1 = 0, \quad (-2,1)$

$\frac{d}{dx}\left[y^2 - 2x - 4y - 1\right] = \frac{d}{dx}[0]$

$2y\frac{dy}{dx} - 2 - 4\frac{dy}{dx} = 0$

$(2y - 4)\frac{dy}{dx} = 2$

$\frac{dy}{dx} = \frac{1}{y - 2}$

At $(-2,1)$, $\frac{dy}{dx} = -1$

a) tangent line:

$y - 1 = -1(x + 2)$

$y = -x - 1$

b) normal line:

$y - 1 = 1(x + 2)$

$y = x + 3$

(33) $2xy + \pi \sin y = 2\pi, \quad (1, \frac{\pi}{2})$

$$\frac{d}{dx}\left[2xy + \pi \sin y\right] = \frac{d}{dx}\left[2\pi\right]$$

$$2\left[x\frac{dy}{dx} + y\right] + \pi(\cos y)\frac{dy}{dx} = 0$$

$$2x\frac{dy}{dx} + 2y + \pi(\cos y)\frac{dy}{dx} = 0$$

$$(2x + \pi\cos y)\frac{dy}{dx} = -2y$$

$$\frac{dy}{dx} = -\frac{2y}{2x + \pi\cos y}$$

At $(1, \frac{\pi}{2})$:

$$\frac{dy}{dx} = \frac{-2(\frac{\pi}{2})}{2(1) + \pi\cos\frac{\pi}{2}} = -\frac{\pi}{2}$$

a) tangent line:

$$y - \frac{\pi}{2} = -\frac{\pi}{2}(x-1)$$

$$y = -\frac{\pi}{2}x + \pi$$

b) normal line:

$$y - \frac{\pi}{2} = \frac{2}{\pi}(x-1)$$

$$y = \frac{2}{\pi}x - \frac{2}{\pi} + \frac{\pi}{2}$$

---

(36) $x^2\cos^2 y - \sin y = 0, \quad (0, \pi)$

$$\frac{d}{dx}\left[x^2\cos^2 y - \sin y\right] = \frac{d}{dx}\left[0\right]$$

$$x^2\left[2\cos y(-\sin y)\frac{dy}{dx}\right] + 2x\cos^2 y - \cos y\frac{dy}{dx} = 0$$

$$-2x^2\sin y\cos y\frac{dy}{dx} - \cos y\frac{dy}{dx} = -2x\cos^2 y$$

$$\frac{dy}{dx} = \frac{2x\cos^2 y}{2x^2\sin y\cos y + \cos y} = \frac{2x\cos y}{2x^2\sin y + 1}$$

At $(0, \pi)$: $\frac{dy}{dx} = \frac{2(0)\cos\pi}{2(0^2)\sin\pi + 1} = 0$

a) tangent line: $y = \pi$

b) normal line: $x = 0$

(39) $x^3y^2 = \cos(\pi y)$

a) $(-1, 1)$

$(-1)^3(1)^2 = \cos(\pi \cdot 1)$

$(-1)(1) = \cos\pi$

$-1 = -1$ ✓

b) $\frac{d}{dx}[x^3y^2] = \frac{d}{dx}[\cos(\pi y)]$

$(x^3)(2y\frac{dy}{dx}) + (y^2)(3x^2) = -\sin(\pi y)\cdot\pi\frac{dy}{dx}$

$2x^3y\frac{dy}{dx} + \pi\sin(\pi y)\frac{dy}{dx} = -3x^2y^2$

$$\frac{dy}{dx} = -\frac{3x^2y^2}{2x^3y + \pi\sin(\pi y)}$$

At $(-1,1)$, $\frac{dy}{dx} = \frac{-3(-1)^2(1)^2}{2(-1)^3(1) + \pi\sin\pi}$

$$\frac{dy}{dx} = \frac{-3}{-2+0} = \frac{3}{2}$$

(42) $x^2 + xy + y^2 = 7$

$\frac{d}{dx}[x^2 + xy + y^2] = \frac{d}{dx}[7]$

$2x + [x\frac{dy}{dx} + y] + 2y\frac{dy}{dx} = 0$

$x\frac{dy}{dx} + 2y\frac{dy}{dx} = -(2x+y)$

$$\frac{dy}{dx} = -\frac{(2x+y)}{x+2y}$$

a) $-\frac{(2x+y)}{x+2y} = 0$

$-(2x+y) = 0 \rightarrow y = -2x$

$x^2 + xy + y^2 = 7$

$x^2 + x(-2x) + (-2x)^2 = 7$

$x^2 - 2x^2 + 4x^2 = 7$

$3x^2 = 7$

$x = \pm\sqrt{\frac{7}{3}}$

$x = \pm\frac{\sqrt{21}}{3}$

$$\boxed{\left(-\frac{\sqrt{21}}{3}, \frac{2\sqrt{21}}{3}\right) \quad \left(\frac{\sqrt{21}}{3}, -\frac{2\sqrt{21}}{3}\right)}$$

b) $-\frac{(2x+y)}{x+2y} = \text{und.}$

$x + 2y = 0$

$x = -2y$

$x^2 + xy + y^2 = 7$

$(-2y)^2 + (-2y)(y) + y^2 = 7$

$3y^2 = 7 \quad y = \pm\frac{\sqrt{7}}{\sqrt{3}} = \pm\frac{\sqrt{21}}{3}$

$$\left(\frac{2\sqrt{21}}{3}, -\frac{\sqrt{21}}{3}\right)$$

$$\left(-\frac{2\sqrt{21}}{3}, \frac{\sqrt{21}}{3}\right)$$

(40) $y^3 - xy = -1$

$$\frac{d}{dx}\left[y^3 - xy\right] = \frac{d}{dx}[-1]$$

$$3y^2\frac{dy}{dx} - \left[x\frac{dy}{dx} + y\right] = 0$$

$$\left(3y^2 - x\right)\frac{dy}{dx} = y$$

$$\frac{dy}{dx} = \frac{y}{3y^2 - x}$$

$$\frac{d^2y}{dx^2} = \frac{(3y^2-x)\frac{dy}{dx} - y\left(6y\frac{dy}{dx} - 1\right)}{(3y^2-x)^2}$$

a) $y^3 - xy = -1$

$y^3 - 2y + 1 = 0$

$y = +1$

$$\begin{array}{r|rrrr} +1 & 1 & 0 & -2 & +1 \\ & & +1 & 1 & -1 \\ \hline & 1 & +1 & -1 & \boxed{0} \end{array}$$

$y^2 + y - 1 = 0$

$$y = \frac{-1 \pm \sqrt{1+4}}{2} = \frac{-1 \pm \sqrt{5}}{2}$$

There are 3 possible y-values for $x = 2$: 1, $\frac{-1+\sqrt{5}}{2}$, $\frac{-1-\sqrt{5}}{2}$

b) At $(2,1)$:

$$f'(2) = \frac{1}{3-2} = 1$$

$$f''(2) = \frac{[3(1^2)-2](1) - (1)[6(1)(1)-1]}{[3(1^2)-2]^2}$$

$$f''(2) = \frac{1-5}{1} = -4$$

$$\boxed{\begin{array}{l} f'(2) = 1 \\ f''(2) = -4 \end{array}}$$

(41) $x^2 + xy + y^2 = 7$

$\frac{d}{dx}[x^2 + xy + y^2] = \frac{d}{dx}[7]$

$2x + [x\frac{dy}{dx} + y] + 2y\frac{dy}{dx} = 0$

$\frac{dy}{dx} = \frac{-(2x+y)}{x+2y}$

Crosses the x-axis ($y=0$):

$x^2 + x(0) + 0^2 = 7$

$x^2 = 7$

$x = \pm\sqrt{7}$

$(-\sqrt{7}, 0)$, $(\sqrt{7}, 0)$

At $(-\sqrt{7}, 0)$: $\frac{dy}{dx} = \frac{2\sqrt{7}}{-\sqrt{7}} = -2$

At $(\sqrt{7}, 0)$: $\frac{dy}{dx} = \frac{-2\sqrt{7}}{\sqrt{7}} = -2$

common slope

$\therefore$ tangents are parallel

---

(43) $2x^2 + 3y^2 = 5$

$\frac{d}{dx}[2x^2 + 3y^2] = \frac{d}{dx}[5]$

$4x + 6y\frac{dy}{dx} = 0$

$\frac{dy}{dx} = -\frac{2x}{3y}$

$y^2 = x^3$

$\frac{d}{dx}[y^2] = \frac{d}{dx}[x^3]$

$2y\frac{dy}{dx} = 3x^2$

$\frac{dy}{dx} = \frac{3x^2}{2y}$

At $(1,1)$: slope of tangent to first curve is $-\frac{2}{3}$
slope of tangent to second curve is $\frac{3}{2}$

At $(1,-1)$: slope of tangent to first curve is $\frac{2}{3}$
slope of tangent to second curve is $-\frac{3}{2}$

In both cases, the tangents are perpendicular. Therefore, the curves are orthogonal at those points.

(45) $v = 8\sqrt{s-t} + 1$

$$\frac{d}{dt}[v] = \frac{d}{dt}\left[8(s-t)^{1/2} + 1\right]$$

$$\frac{dv}{dt} = 4(s-t)^{-1/2}\left[\frac{ds}{dt} - 1\right]$$

$$\frac{dv}{dt} = \frac{4(v-1)}{\sqrt{s-t}} = \frac{4\left[(8\sqrt{s-t}+1)-1\right]}{\sqrt{s-t}}$$

$$\frac{dv}{dt} = \frac{32\sqrt{s-t}}{\sqrt{s-t}} = 32\ ft/sec^2$$

---

(46) $y^4 - 4y^2 = x^4 - 9x^2$

$$\frac{d}{dx}\left[y^4 - 4y^2\right] = \frac{d}{dx}\left[x^4 - 9x^2\right]$$

$$4y^3\frac{dy}{dx} - 8y\frac{dy}{dx} = 4x^3 - 18x$$

$$\frac{dy}{dx} = \frac{4x^3 - 18x}{4y^3 - 8y} = \frac{2x^3 - 9x}{2y^3 - 4y}$$

At $(-3, 2)$, $\frac{dy}{dx} = \frac{-54+27}{16-8} = -\frac{27}{8}$

At $(-3, -2)$, $\frac{dy}{dx} = \frac{-54+27}{-16+8} = \frac{27}{8}$

At $(3, 2)$, $\frac{dy}{dx} = \frac{54-27}{16-8} = \frac{27}{8}$

At $(3, -2)$, $\frac{dy}{dx} = \frac{54-27}{-16+8} = -\frac{27}{8}$

(48) $x^2+2xy-3y^2=0$

$\frac{d}{dx}\left[x^2+2xy-3y^2\right]=\frac{d}{dx}[0]$

$2x+2\left[x\frac{dy}{dx}+y\right]-6y\frac{dy}{dx}=0$

$2x+2x\frac{dy}{dx}+2y-6y\frac{dy}{dx}=0$

$(2x-6y)\frac{dy}{dx}=-(2x+2y)$

$\frac{dy}{dx}=\frac{-(x+y)}{x-3y}=\frac{x+y}{3y-x}$

At $(1,1)$, $\frac{dy}{dx}=\frac{2}{2}=1$

Normal line:

$y-1=-1(x-1)$

$y=-x+2$

$x^2+2xy-3y^2=0$

$x^2+2x(-x+2)-3(-x+2)^2=0$

$x^2-2x^2+4x-3x^2+12x-12=0$

$-4x^2+16x-12=0$

$x^2-4x+3=0$

$(x-1)(x-3)=0$

$x=1$ or $x=3$

$y=-x+2$

$y=-3+2$

$y=-1$

$\boxed{(3,-1)}$

(47) not assigned

$x^3+y^3-9xy=0$

$\frac{d}{dx}[x^3+y^3-9xy]=\frac{d}{dx}[0]$

$3x^2+3y^2\frac{dy}{dx}-9[x\frac{dy}{dx}+y]=0$

$3y^2\frac{dy}{dx}-9x\frac{dy}{dx}=9y-3x^2$

$\frac{dy}{dx}=\frac{9y-3x^2}{3y^2-9x}=\frac{3y-x^2}{y^2-3x}$

a) At $(4,2)$, $\frac{dy}{dx}=\frac{3(2)-4^2}{2^2-3(4)}$

$=\frac{6-16}{4-12}=\frac{-10}{-8}=\frac{5}{4}$

At $(2,4)$, $\frac{dy}{dx}=\frac{3(4)-2^2}{4^2-3(2)}$

$=\frac{12-4}{16-6}=\frac{4}{5}$

b) $\frac{3y-x^2}{y^2-3x}=0$

$3y-x^2=0$

$3y=x^2$

$y=\frac{x^2}{3}$

$x^3+y^3-9xy=0$

$x^3+\left(\frac{x^2}{3}\right)^3-9x\left(\frac{x^2}{3}\right)=0$

$x^3+\frac{x^6}{27}-3x^3=0$

$\frac{x^6}{27}-2x^3=0$

$x^6-54x^3=0$

$x^3(x^3-54)=0$

$x^3=54$ or $x=0$

$x=3\sqrt[3]{2}$

$\boxed{\left(3\sqrt[3]{2},\ 3\sqrt[3]{4}\right)}$

$y=\frac{(3\sqrt[3]{2})^2}{3}=\frac{9\sqrt[3]{4}}{3}$

c) $\frac{3y-x^2}{y^2-3x}=$ und

$y^2-3x=0$

$x=\frac{y^2}{3}$

$x^3+y^3-9xy=0$

$\left(\frac{y^2}{3}\right)^3+y^3-9\left(\frac{y^2}{3}\right)(y)=0$

$\frac{y^6}{27}+y^3-3y^3=0$

$y^6-54y^3=0$

$y^3(y^3-54)=0$

$y=0$ or $y=3\sqrt[3]{2}$

$x=\frac{(3\sqrt[3]{2})^2}{3}$

$x=\frac{9\sqrt[3]{4}}{3}$

$\boxed{\left(3\sqrt[3]{4},\ 3\sqrt[3]{2}\right)}$

(49) not assigned

$xy + 2x - y = 0$

$\frac{d}{dx}[xy + 2x - y] = \frac{d}{dx}[0]$

$\left[x\frac{dy}{dx} + y\right] + 2 - \frac{dy}{dx} = 0$

$x\frac{dy}{dx} - \frac{dy}{dx} = -(y+2)$

$\frac{dy}{dx} = -\frac{(y+2)}{x-1} = \frac{y+2}{1-x}$

$2x + y = 0$

$m = -2$

$\frac{y+2}{1-x} = \frac{1}{2}$

$2y + 4 = 1 - x$

$x = -2y - 3$

$xy + 2x - y = 0$

$(-2y-3)y + 2(-2y-3) - y = 0$

$-2y^2 - 3y - 4y - 6 - y = 0$

$-2y^2 - 8y - 6 = 0$

$y^2 + 4y + 3 = 0$

$(y+3)(y+1) = 0$

$y = -3$ or $y = -1$

$(3, -3)$

$(-1, -1)$

At $(3, -3)$:

$y + 3 = -2(x - 3)$

$y = -2x + 3$

At $(-1, -1)$:

$y + 1 = -2(x + 1)$

$y = -2x - 3$

(50) $x = y^2$

$$\frac{d}{dx}[x] = \frac{d}{dx}[y^2]$$

$$1 = 2y\frac{dy}{dx}$$

$$\frac{dy}{dx} = \frac{1}{2y} \rightarrow \text{Slope of tangent line}$$

$$-2y \rightarrow \text{slope of normal line}$$

At (b^2, b), normal line is:

$$y - b = -2b(x - b^2)$$

$$y = -2bx + 2b^3 + b$$

This line intersects the x-axis at:

$$0 = -2bx + 2b^3 + b$$

$$x = \frac{2b^3 + b}{2b} = b^2 + \frac{1}{2},$$

which is the value of a and must be greater than $\frac{1}{2}$ if $b \neq 0$

The two normals at $(b^2, \pm b)$ will be perpendicular when they have slopes of -1 and 1, which gives

$$-2y = -1 \quad \text{or} \quad -2y = 1$$

$$y = \frac{1}{2} \quad \text{or} \quad y = \frac{-1}{2}$$

(or $b = \pm\frac{1}{2}$).

The corresponding value of a is

$$x = b^2 + \frac{1}{2}$$

$$a = \left(\frac{1}{2}\right)^2 + \frac{1}{2} = \frac{3}{4}.$$

Thus, the two nonhorizontal normals are perpendicular when $a = \frac{3}{4}$.